Freie Universität Berlin
Tutorials on Quantum Information Theory
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Problem Sheet 2
Foundations

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Remark: *We will also discuss the parts of the previous exercise sheet that have not yet been discussed. So in case you have not already prepared it, this is your chance.*

1. **The most general quantum measurements.**

In a quantum mechanics course, measurements are typically introduced as projective measurements of the eigenvalues of observables. But from a theoretical perspective another measurement description is often helpful. For simplicity—and in the spirit of information theory—we assume that the possible measurement outcomes are from a discrete set $\mathcal{X}$.[1]

A measurement with outcomes $\mathcal{X}$ on a quantum system with Hilbert space $\mathcal{H}$ can be described by a *positive operator valued measure* (POVM) on $\mathcal{X}$. We denote by $\mathrm{Pos}(\mathcal{H}) := \{A \in L(\mathcal{H}) \mid A \succcurlyeq 0\}$ the set of Hermitian positive semi-definite operators on $\mathcal{H}$. A POVM on a discrete space $\mathcal{X}$ is a map $\mu : \mathcal{X} \to \mathrm{Pos}(\mathcal{H})$ such that $\sum_{x\in\mathcal{X}} \mu(x) = \mathrm{Id}$. If the system is in the quantum state $\rho \in \mathcal{D}(\mathcal{H})$, the probability of observing the outcome $x \in \mathcal{X}$ is given by $\mathrm{Tr}(\mu(x)\rho)$.

a) What is the difference between POVM measurements and the measurement description using observables?

Solution: Let $A = \sum_i \lambda_i \Pi_i$ be an observable with $\mathrm{spec}(A) = \{\lambda_i\}$ and Π_i the orthogonal projector to the i-th eigenspace. Then, the map $\mathrm{spec}(A) \to \mathrm{Pos}(\mathcal{H})$, $\lambda_i \mapsto \Pi_i$ defines a POVM, because $\sum_i \Pi = \mathrm{Id}$. The converse however the constituent operators $\mathrm{range}(\mu) = \{E_i\}$ of a POVM μ are not required to be orthogonal projectors, i.e. in general we do not have $E_iE_j = \delta_{ij}E_j$ as for the so-called projector valued measurements (PVM) that can be directly expressed as observables. Nevertheless every POVM can be implemented with PVMs using an ancillar system. More on this, probabliy on a up-coming sheet.

It is often stated that this is the most general form of a quantum measurement. We want to understand this statement in more detail. So what could be regarded as the most general quantum measurement? One can start as follows: A (general) quantum measurement M with outcomes in $\mathcal{X}$ is a map that associates to each quantum state $\rho \in \mathcal{D}(\mathcal{H})$ a probability measure p_ρ on $\mathcal{X}$, i.e. $M : \rho \mapsto p_\rho$ with $p_\rho : \mathcal{X} \to [0, 1]$ such that $\sum_{x\in X} p_\rho(x) = 1$.

b) Show that there is a one-to-one mapping between general quantum measurements as defined above and POVMs on $\mathcal{X}$.

Solution: Let M be a general measurement. To make sense of the other principles of quantum mechanics, in particular the statistical interpration mixtures of quantum states, we have to require that M is a linear map (It might have been a good idea to already put this in the definition, but I forgot this).

[1] More generally, one can replace $\mathcal{X}$ by the σ-algebra of a measurable Borel space. This is the natural structure from probability theory to describe a set of all possible events in an experiment. If you are curious and have some time left, it is an instructive and not so hard exercise to look up the definitions of a Borel space and a probability space and translate this exercise and its solution into this language.

Then, for fixed $x \in \mathcal{X}$ the map $\rho \mapsto p_\rho(x)$ is by definition an arbitrary element of the dual space of $\mathcal{D}(\mathcal{H})$. Being equipped with an inner product, we can use the canonical isomorphism $L(\mathcal{H}) \simeq L^*(\mathcal{H})$ to express every element in the dual space as an element in $L(\mathcal{H})$. Explicitely, we can define $\mu(x) \in L(\mathcal{H})$ such that $\rho \mapsto p_\rho(x) = (\mu(x), \rho)$. The restriction to $p_\rho(x) \geq 0$ for all ρ and x amounts to restricting $\mu(x)$ to an positive semi-definite operator. (Recall that $\mathrm{Tr}(A\rho) \geq 0$ for all $\rho \in \mathcal{D}(\mathcal{H})$ if and only if $A \succcurlyeq 0$. To see this express the trace in the eigenbasis of ρ or A.)

Now, for fixed ρ if $x \mapsto p_\rho(x)$ should define a probability measure, we have the restriction that $\sum_{x \in X} p_\rho(x) = \sum_{x \in X} (\mu(x), \rho) = 1$ for all ρ. This is the case if and only if $\sum_{x \in X} \mu(x) = \mathrm{Id}$ (Uniqueness can be seen e.g. by parameter counting).

Can you come up with a more general notion of quantum measurements?

Solution: I can not.

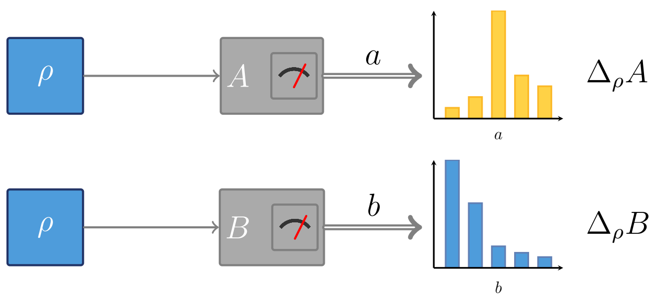
2. **Impossible machines – on joint measurability.**

As we all know from Kindergarten, in quantum mechanics position and momentum, and in fact every pair of incompatible observables, cannot be jointly measured on the same state with 100 % precision each. Or this is what they tell us Heisenberg's uncertainty relation quantifies ...

In fact, **?** himself seems to have had something like this in mind, when he described a setting in which one tries to measure the position of a particle very precisely through a microscope, arguing that photons hitting the particle to be measured will impart on it an indeterminate Compton recoil on the order of h/λ, whence the uncertainty relation $\Delta x \, \Delta p \sim \lambda \cdot h/\lambda = h$. This is sufficiently imprecise and non-rigorous to admit many different interpretations.

Therefore, in this problem, we will take a closer look at different variants of formulating Heisenberg-type uncertainty relations.

Preparation uncertainty. The first type of uncertainty relations deals with *preparations* of individual states. An uncertainty relation of this type quantifies the impossibility of preparing states on which two incompatible observables A and B both have definite values. So we are given many copies of a state ρ and want to determine $\mathrm{Tr}[\rho A]$ as well as $\mathrm{Tr}[\rho B]$ by measuring the two observables on separate copies of ρ. The setting for preparation uncertainty is the following:



A preparation-uncertainty relation quantifies the product of the variances $\Delta_\rho A := (A - \langle A \rangle_\rho)^2$ and $\Delta_\rho B := (B - \langle B \rangle_\rho)^2$.

a) Use the Cauchy-Schwarz inequality to derive the lower bound

$$\Delta_\rho A \, \Delta_\rho B \geq \frac{1}{4} |\langle [A, B] \rangle_\rho|^2$$

Solution: Choose a vector $|\psi\rangle$ and define two operators $A' = (A - \langle A\rangle)$, and $B' = (B - \langle B\rangle)$. Then

$$\frac{1}{4}|\langle[A,B]\rangle_\rho|^2 = |\mathrm{Im}\,\langle\psi|\,A'B'\,|\psi\rangle\,|^2 \leq |\,\langle\psi|\,A'B'\,|\psi\rangle\,|^2$$
$$\leq \|A'\,|\psi\rangle\,\|^2\|B'\,|\psi\rangle\,\|^2 = \langle A'^2\rangle_\psi\langle B'^2\rangle_\psi$$

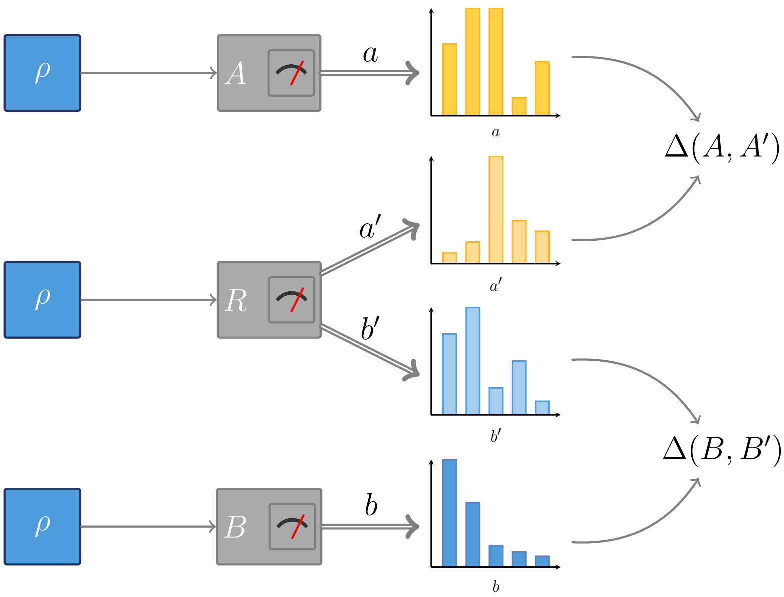
Measurement uncertainty: On the other hand, one can also derive measurement uncertainty relations, or so called error-disturbance relations. In this case, we are given a single state and want to infer the value of two observables A and B by measuring the two on the very same state. One might imagine two ways of doing so, one is *sequential measurement* of A and B. The other is to perform some two-outcome measurement R such that the marginals A' and B' of R can be used to approximate the outcome distribution of A and B. This is what we will refer to as *joint measurement* and is actually a very recent topic of research (Busch et al., 2013, 2014b; Ozawa, 2004).

It is important to note that in contrast to preparation uncertainty relations measurement uncertainty relations are state-independent. They give us guarantees on how well measurement devices can function on arbitrary input states.

We can formalise this setting in the language of POVMs as introduced in Problem 2 as follows: Let A and B be two POVMs with outcome sets X and Y. Two every element $x \in X$ we can thus associate a positive semi-definite operator $A(x)$ such that $\mathrm{Tr}[A(x)\rho]$ is the probability of obtaining outcome x when measuring A, and likewise for B. In slight abuse of notation[2], for any subset $M \subset X$ we can also define $A(M) = \sum_{x\in M} A(x)$. A joint measurement is then a two-outcome POVM R on the outcome set $X \times Y$, whose marginals are given as

$$A'(x) := \sum_{y\in Y} R(x,y), \quad B'(x) := \sum_{x\in X} R(x,y)\,,$$

so by simply ignoring the respective other outcomes.



We will start by deriving some simple statements about conditions for joint measurability.

b) Show that if $A(x)$ and $B(y)$ commute for all $x \in X, y \in Y$, joint measurement is possible.

[2]This notation is actually natural if the outcome sets are σ-algebras.

Solution: Claim: $R(x,y) \coloneqq A(x)B(y)$ does the job.

Proof:

i. $R(x,y) \geq 0$: Since $[A(x),B(y)] = 0$ we can jointly diagonalise them with a unitary U such that $A(x)B(y) = UD_{A(x)}U^\dagger UD_{B(y)}U^\dagger = UD_{A(x)}D_{B(y)}U^\dagger \geq 0$ since $A(x), B(y) \geq 0$ and the eigenvalues are multiplied.

ii. $\sum_x R(x,y) = B(y)$ and $\sum_y R(x,y) = A(x)$:

$$\sum_x R(x,y) = \sum_x A(x)B(y) = B(y)\sum_x A(x) = B(y) \cdot \mathbb{1},$$

since A is a POVM. Likewise for B.

c) Let O be a positive semi-definite operator and P be a projection. Show that $P \succcurlyeq O \succcurlyeq 0$ implies $O = OP = PO = POP$. (Here we use the notation that two operators O, Q fulfil $O \succcurlyeq Q$ if $O - Q$ is positive semi-definite.)

Solution: We have

$$0 \leq (1-P)O(1-P) \leq (1-P)P(1-P) = 0$$

since for any X, if $O \geq 0$ then $XOX^\dagger \geq 0$. We can then write $X^\dagger X = (1-P)O(1-P)$ with $X = \sqrt{O}(1-P)$. We then have $\sqrt{O}X = O(1-P) = 0$ and hence $O = OP = O^\dagger = (OP)^\dagger = POP$

d) Show that the condition in (b) is also necessary for joint measurement if A is a projective measurement, that is, $A(x)$ is an orthogonal projection for all $x \in X$ and $A(x)A(y) = \delta_{xy}A(x)$.

Hint: Express a joint measurement $R(x,y)$ as a sum of the observables $R(x,y)$ and $A(x)R(\{x\}^c, y)$ and use the relation derived in (c).

Solution: Observe that $R(x,y) = A(x)R(x,y)$ by (c), and likewise $R(\{x\}^c, y) = A(\{x\}^c, y)R(\{x\}^c, y)$. But $A(x)A(\{x\}^c) = 0$ since A is an orthogonal projection.

We then have

$$\begin{aligned} R(x,y) &= A(x)R(x,y) + A(x)A(\{x\}^c)R(\{x\}^c, y) && (1)\\ &= A(x)(R(x,y) + R(\{x\}^c, y)) && (2)\\ &= A(x)B(y), && (3) \end{aligned}$$

and since $R(x,y) \geq 0$ we have $A(x)B(y) = (A(x)B(y))^\dagger = B(y)A(x)$, so A and B commute.

To actually derive an uncertainty relation quantifying the best possible joint measurement of incompatible observables, of course, one first has to spell out an appropriate distance measure.

One such choice is the state-dependent Wasserstein distance

$$\Delta(A_\rho, A'_\rho) = \left[\inf_\gamma \iint (x-x')^2 \mathrm{d}\gamma(x,x')\right]^{\frac{1}{2}},$$

where the infimum is taken over all couplings γ of A_ρ and A'_ρ, that is, all joint distributions with marginals A_ρ, A'_ρ. One can now take

$$\Delta(A, A') \coloneqq \sup_\rho \Delta(A_\rho, A'_\rho),$$

as the worst-case measure over all states.

Now consider two qubit observables $E : \pm \mapsto E_\pm$ and $F : \pm \mapsto F_\pm$ with POVM elements $E_\pm = \frac{1}{2}(e_0 \mathbb{1} \pm \mathbf{e}\cdot\sigma)$ and $F_\pm = \frac{1}{2}(f_0 \mathbb{1} \pm \mathbf{f}\cdot\sigma)$, as well as a state $\rho = \frac{1}{2}(\mathbb{1} + \mathbf{r}\cdot\sigma)$, where $\mathbf{e}, \mathbf{f}, \mathbf{r}$ are unit vectors and $\sigma = (X, Y, Z)$ denotes the vector of all Pauli matrices. Finally, denote by $E_\rho = (\mathrm{Tr}[E_-\rho], \mathrm{Tr}[E_+\rho])$ and $F_\rho = (\mathrm{Tr}[F_-\rho], \mathrm{Tr}[F_+\rho])$ the corresponding probability distributions.

e) Show that

$$\gamma(+,+) \equiv \gamma, \quad \gamma(+,-) = E_\rho(+) - \gamma$$
$$\gamma(-,+) = F_\rho(+) - \gamma, \quad \gamma(-,-) = 1 - E_\rho(+) - F_\rho(+) + \gamma$$

is a coupling of the type above, and argue that it is the most general one.

Solution: Check the relations

$$\gamma(+,+) + \gamma(+,-) = E_\rho(+) \tag{4}$$
$$\gamma(-,+) + \gamma(-,-) = 1 - E_\rho(+) \tag{5}$$
$$\gamma(+,+) + \gamma(-,+) = F_\rho(+) \tag{6}$$
$$\gamma(+,-) + \gamma(-,-) = 1 - F_\rho(+). \tag{7}$$

There are three constraints, so we have one free variable (γ).

f) Determine $\Delta(E_\rho, F_\rho)$ as well as $\Delta(E, F)$.

Solution: We find

$$\Delta(E_\rho, F_\rho)^2 = \min_\gamma 4(E_\rho(+) - \gamma) + 4(F_\rho(+) - \gamma), \tag{8}$$

since the other two terms vanish.

The minimum is attained at $\gamma = \min\{E_\rho(+), F_\rho(+)\}$ by the positivity constraint on γ ($\gamma(\pm, \mp) \geq 0$).

We can determine $E_\rho(+)$ to be

$$E_\rho(+) = \frac{1}{4}(\mathrm{Tr}(e_0\mathbb{1}) + \mathrm{Tr}(\mathbf{e}\cdot\sigma) + \mathrm{Tr}(\mathbf{r}\cdot\sigma) + \mathrm{Tr}(\mathbf{e}\cdot\sigma)(\mathbf{r}\cdot\sigma))) \tag{9}$$
$$= \frac{1}{2}(e_0 + \mathbf{e}\cdot\mathbf{r}), \tag{10}$$

since the Pauli matrices are traceless and square to the identity.

Then we get if, without loss of generality, $E_\rho(+) \leq F_\rho(+)$

$$\Delta(E_\rho, F_\rho)^2 = 2(f_0 - e_0 + \mathbf{r}\cdot(\mathbf{f} - \mathbf{e})). \tag{11}$$

One can now show (Busch et al., 2014a) the measurement uncertainty relation

$$\Delta(A, A_R)^2 + \Delta(B, B_R)^2 \geq \sqrt{2}\left[\|\mathbf{a} - \mathbf{b}\| + \|\mathbf{a} + \mathbf{b}\| - 2\right],$$

where A_R, B_R denote the respective marginals of a joint measurement R.

References

Busch, P., P. Lahti, and R. F. Werner (2013, October). Proof of Heisenberg's Error-Disturbance Relation. *Physical Review Letters 111*(16).

Busch, P., P. Lahti, and R. F. Werner (2014a, January). Heisenberg uncertainty for qubit measurements. *Physical Review A 89*(1).

Busch, P., P. Lahti, and R. F. Werner (2014b, December). *Colloquium* : Quantum root-mean-square error and measurement uncertainty relations. *Reviews of Modern Physics 86*(4), 1261–1281.

Ozawa, M. (2004, June). Uncertainty relations for noise and disturbance in generalized quantum measurements. *Annals of Physics 311*(2), 350–416.