**Problem Sheet** 3
**POVM, impossibilities and $\ell_p$-norms**

1. **Encoding classical bits.** On the last excercise sheet we introduced the description of quantum measurements with the help of POVMs. We want to use this formulation to study the following question:

   Let $\mathcal{H}$ be a $d$-dimensional Hilbert space. Our aim is to encode $n$ classical bits into the space of quantum states $\mathcal{D}(\mathcal{H})$. To this end, we choose a set of $2^n$ states $\{\rho_i\}_{i\in\{0,1\}^n} \subset \mathcal{D}(\mathcal{H})$, each state corresponding to a bit string. To decode the bit string we have to make a measurement described by a POVM $\{F_i\}_{i\in\{0,1\}^n\}}$, where the bit string is the outcome.

   How many classical bits can be encoded and decoded in a $d$-dimensional quantum system in this way?

   Consider a source that outputs the bit string $x \in \{0,1\}^n$ with probability $p(x)$.

   a) Define the success probability of the decoding procedure.

   b) Show that for $p(x) = 2^{-n}$ the success probability is bounded by $2^{-n}d$.
   (*Hint:* Argue that $\mathbb{1} \succcurlyeq \rho_i$ for all $i$ and show that for $A \succcurlyeq 0$ and $B \succcurlyeq C$ it holds that $\mathrm{Tr}(AB) \geq \mathrm{Tr}(AC)$ as a starting point.)

   c) What does this imply?

2. **Impossible machines – no cloning.**

   In this problem we will re-derive the impossibility results that you have seen in the lecture but now directly using the structure of quantum theory.

   Show that there does not exist a unitary map on two copies of a Hilbert space $\mathcal{H}$ which acts in the following way:

   $$\forall \, |\psi\rangle \in \mathcal{H} : \, U \, |\psi\rangle \, |0\rangle = \mathrm{e}^{\mathrm{i}\phi(\psi)} \, |\psi\rangle \, |\psi\rangle \; .$$

3. **$\ell_p$-norms**
   In quantum information we deal with a handful of different matrix spaces such as the set of quantum states and in the near future also quantum channels. For quantitative statements we have to equip these spaces with distance measures. Depending on the application and context different distance measures have the desired operational meaning.

   A prominent role is played by the so called *Schatten p-norms*. But to set the stage we have to first familiarise ourself with their analogons on vector spaces, namely $\ell_p$-*norms*. For $1 \leq p < \infty$ the $\ell_p$-norm on the complex vector space $\mathbb{C}^n$ is defined as

   $$\| \cdot \|_{\ell_p} : x \mapsto \|x\|_{\ell_p} := \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}},$$

   and the $\ell_\infty$-norm as

   $$\| \cdot \|_{\ell_\infty} : x \mapsto \|x\|_{\ell_\infty} := \lim_{p\to\infty} \|x\|_{\ell_p}.$$

   We will now characterise the function $\| \cdot \|_{\ell_p}$ and derive important properties. We begin with an explicit expression for the $\ell_\infty$-norm.

a) Show that $\|x\|_{\ell_\infty} = \max_{1 \leq i \leq n} |x_i|$.

For all of what follows the notion of a convex function will be important. Let $D \subset \mathbb{R}$ be a convex set. We say that a function $f : D \to \mathbb{R}$ is *convex* if

$$f\left(\sum_i a_i x_i\right) \leq \sum_i a_i f(x_i),$$

for all $x_i \in D$ and $a_i \geq 0, i = 1, \ldots, m$ such that $\sum_i a_i = 1$.

b) Show that any twice continuously differentiable function on an open intervall is convex if and only if its second derivative is everywhere nonnegative.

c) Show that $|\cdot|^p$ is a convex function for $p \geq 1$.

We will now use this fact to show that $\|\cdot\|_{\ell_p}$ is a norm (positive definite, absolutely homogeneous, subadditive aka triangle inequality).

d) Argue that $\|\cdot\|_{\ell_p}$ is positive definite and absolutely homogeneous for $1 \leq p < \infty$ and $p = \infty$.

That was easy. Now comes the hard part; we have to show that the norms satisfy the triangle inequality, i.e.

$$\|x + y\|_{\ell_p} \leq \|x\|_{\ell_p} + \|y\|_{\ell_p}. \tag{1}$$

In fact, the triangle inequality for $\ell_p$-norms has even its own name, *Minkowski inequality*. A clever way to prove this inequality is to normalise the right hand side, introduce normalised vectors and then use the convexity of $|\cdot|^p$.

e) Argue that it is sufficient to consider the case $\|x\|_{\ell_p} = \lambda$ and $\|y\|_{\ell_p} = (1 - \lambda)$ with $\lambda \in (0, 1)$ in order to proof the Minkowski inequality.

f) Show the Minkowski inequality for the $\ell_p$-norms when $1 \leq p < \infty$.

A crucial property of the $\ell_p$-norms is Hölder's inequality. It generalises the Cauchy-Schwarz inequality, which is its special case for $p = 2$. Let $\langle \cdot, \cdot \rangle$ be the Euclidean inner product on $\mathbb{C}^n$, i.e. $\langle x, y \rangle = \sum_{i=1}^n \bar{x}_i y_i$ with $\bar{\cdot}$ denoting the complex conjugate. Hölder's inequality reads

$$|\langle x, y \rangle| \leq \|x\|_{\ell_p} \|y\|_{\ell_q}, \text{ where } \frac{1}{p} + \frac{1}{q} = 1.$$

Like for the proof of Minkowski's inequality, it will be useful to use normalised vectors in the proof of Hölder's inequality. Furthermore, we will need to first establish the *arithmetic-geometric mean inequality*

$$\prod_{i=1}^n x_i^{a_i} \leq \sum_{i=1}^n a_i x_i \text{ if } x_i \geq 0, \, a_i \geq 0, \, \sum_i a_i = 1. \tag{2}$$

g) Show that $-\log$ is a convex function and use this to show the arithmetic-geometric mean inequality, Eq. (2).

h) Now, prove Hölder's inequality for $1 < p < \infty$.

i) Finally, prove Hölder's inequality for $p = 1$.

More generally, for a norm $\|\cdot\|$ on $\mathbb{C}^d$ one can define its dual norm $\|\cdot\|^*$ as

$$\|x\|^* := \sup_{x \in \mathbb{C}^d, \|y\|=1} |\langle x, y \rangle|. \tag{3}$$

j) Show that for every norm $\| \cdot \|$ on $\mathbb{C}^d$ it holds:

$$|\langle x, y \rangle| \leq \|x\| \|y\|^* \tag{4}$$

for all $x, y \in \mathbb{C}^d$.

k) Show that for the dual norm $\| \cdot \|_{\ell_p}^*$ of the $\ell_p$-norm $\| \cdot \|_{\ell_p}$ is the norm $\| \cdot \|_{\ell_q}$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Finally, we will show another convenient property of the $\ell_p$ norms.

l) Show that the $\ell_p$ norms are ordered in the sense that

$$\|x\|_{\ell_p} \leq \|x\|_{\ell_q}, \text{ for } q \leq p.$$

4. **Non-uniqueness of the decomposition of mixed states.**

Consider two macroscopically different preparation schemes of a large number of polarised photons:

**Preparation A.** For each photon we toss a fair coin. Depending on whether we get head or tail, we prepare the photon to have either vertical or horizontal *linear* polarisation.

**Preparation B.** For each photon we toss a fair coin. Depending on whether we get head or tail, we prepare the photon to have either left-handed or right-handed *circular* polarisation.

We are given a large number of photons which all were prepared by the same scheme.

a) Argue that having only access to the photons we can not distinguish which of the preparation scheme was used.

b) Argue that if it were possible to distinguish such types of preparations by measuring the photon, locality would be violated.