Freie Universität Berlin

Tutorials on Quantum Information Theory

Summer Term 2018

Problem Sheet 4

Teleportation, Schmidt decomposition and Schatten norms

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1. **Schmidt decomposition and purification**

In the lecture, you already saw the Schmidt decomposition of bipartite quantum states $|\Psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ as given by

$$|\Psi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_j} \left|\psi_j^1\right\rangle \left|\psi_j^2\right\rangle,$$

where $\{\left|\psi_j^i\right\rangle\}$ are orthonormal bases of $\mathcal{H}_i$.

In this exercise, we will study some useful properties and applications of the Schmidt decomposition. To begin with, let us look at states with the same Schmidt coefficients, that is

$$|\Psi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_j} \left|\psi_j^1\right\rangle \left|\psi_j^2\right\rangle, \quad |\Phi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_j} \left|\phi_j^1\right\rangle \left|\phi_j^2\right\rangle.$$

a) Show that $|\Psi\rangle$ and $|\Phi\rangle$ are related by a local unitary, i.e., a unitary of the form $U \otimes V$ with U and V unitary. Give that unitary explicitly.

b) Show that any local unitary transformation leaves the Schmidt coefficients invariant.

This gives rise to a nice interpretation of the Schmidt coefficients of a state in terms of entanglement (more soon!):

c) Determine the reduced density matrices $\rho_1 = \mathrm{Tr}_2 |\Psi\rangle\langle\Psi|$ and $\rho_2 = \mathrm{Tr}_1 |\Psi\rangle\langle\Psi|$. How can the Schmidt coefficients be interpreted? What are the Schmidt coefficients of the maximally entangled state?

d) Use the Schmidt decomposition to show that *any* bipartite state $|\Psi\rangle$ can be expressed as

$$|\Psi\rangle = (X \otimes \mathbb{1}) |\Omega\rangle,$$

where $|\Omega\rangle$ is a maximally entangled state.

The maximally entangled state is *invariant* under certain product unitaries $U \otimes V$.

e) What are the conditions on U and V for this to be the case?

Recall from the lecture that for any quantum state $\rho \in \mathcal{L}(\mathcal{H})$ there exists a pure quantum state $|\psi_\rho\rangle \in \mathcal{H} \otimes \mathcal{G}$ such that $\mathrm{Tr}_{\mathcal{G}}[|\psi_\rho\rangle\langle\psi_\rho|] = \rho$. The Schmidt decomposition is useful for explicitly constructing such purifications:

f) Give a purification of an arbitrary quantum state ρ in terms of its eigenvalues and eigenvectors.

g) Show that two purifications $|\psi_1^\rho\rangle$ and $|\psi_2^\rho\rangle$ of the same state ρ are related by a unitary transformation that acts on $\mathcal{G}$ only.

2. **General teleportation schemes**

In the lecture you saw a teleportation scheme using a maximally entangled state shared by Alice and Bob. In this exercise we will generalise this setting to teleportation schemes with higher local dimensions.

We begin by reformulating the qubit teleportation scheme in terms of Bell-basis measurements. The Bell basis for two qubits is given by

$$|\Phi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), |\Phi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),$$
$$|\Phi_2\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), |\Phi_3\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

a) Show that the Bell basis can be prepared starting from $|\Phi_0\rangle$ using local Pauli operations only.

b) Show that the scheme from the lecture is equivalent to the following one:

Alice and Bob share a maximally entangled state $|\Phi_0\rangle$, Alice prepares a state $|\omega\rangle = \alpha|0\rangle + \beta|1\rangle$, measures in the Bell basis and transmits her measurement result to Bob who applies the corresponding Pauli operator.

This reformulation generalises to a d-dimensional teleportation scheme in which Alice and Bob share a maximally entangled state $|\omega\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d}|ii\rangle$. As above the scheme is based on measuring in a maximally entangled orthonormal basis set $\{|\Psi_\alpha\rangle\}_{\alpha=1}^{d^2}$, i.e., an orthonormal basis for which $\mathrm{Tr}_1[|\Psi_\alpha\rangle\langle\Psi_\alpha|] = \mathbb{1}_d = \mathrm{Tr}_2[|\Psi_\alpha\rangle\langle\Psi_\alpha|]$.

There exist several constructions of linearly independent sets $\{U^\alpha\}_{\alpha=1}^{d^2}$ of d^2 trace-wise orthogonal unitary operator $U^\alpha \in U(d)$,

$$\mathrm{Tr}[U^{\alpha\dagger}U^\beta] = d\delta_{\alpha\beta}$$

for all α and β. In the following, we just assume the existence of such a set.

c) Show that such a set $\{U^\alpha\}_{\alpha=1}^{d^2}$ gives rise to a maximally entangled basis set by setting

$$|\Psi_\alpha\rangle = U^\alpha \otimes \mathbb{1}|\omega\rangle .$$

d) Use the completeness relation for $\{|\Psi_\alpha\rangle\}_\alpha$ to show that any such operator basis satisfies

$$\frac{1}{d}\sum_\alpha U^\alpha_{ij}\overline{U}^\alpha_{kl} = \delta_{ik}\delta jl. \tag{1}$$

e) Expand the basis states $|\Psi_\alpha\rangle$ in the computational product basis $\{|ij\rangle\}_{ij}$ and give the basis transformation Γ and its inverse Γ^{-1} explicitly. *Hint: Identify the indices α and i, j suitably.*

f) Check the unitarity of Γ using Eq. (1) and express the states $|ij\rangle$ in the basis $|\Psi_\alpha\rangle$.

Now consider the setting in which Alice and Bob share the state $|\omega\rangle$ and Alice measures her part of the system in the basis $|\Psi_\alpha\rangle$.

g) Insert the resolution of the identity $\sum_{ij}|ij\rangle\langle ij| = \sum_{\alpha,ij}(\Gamma^{-1})^\alpha_{ij}|\Psi_\alpha\rangle\langle ij|$ to derive the unitary corrections required in the d-dimensional teleportation scheme.

Further reading:

Bennett et al. (1993): The original teleportation paper.

Banaszek (2000): A d-dimensional teleportation scheme.

3. **Schatten p-norms**

On the last excercise sheet we have studied the ℓ_p-norms on vector spaces. The ℓ_p-norms have important cousins on matrix spaces, the Schatten p-norms. As they are important distant measures in quantum information, we study there different definitions and properties in this excerice.

One way to introduce the Schatten p-norm with $p \in [1, \infty)$ for a matrix $A \in \mathbb{C}^{n\times n}$ is

$$\|A\|_p := \left(\operatorname{Tr}[|A|^p]\right)^{\frac{1}{p}}, \tag{2}$$

where $|A| := \sqrt{A^\dagger A}$ is the matrix absolute value. Furthermore, the case $p = \infty$ is defined as the limit $\|A\|_\infty = \lim_{p\to\infty} \|A\|_p$.

These norms are related to the ℓ_p-norms of the eigenvalues (or more generally the singular values) of A.

a) Let A be a Hermitian matrix and let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be the vector of its eigenvalues. Show that

$$\|A\|_p = \|\lambda\|_{\ell_p} \tag{3}$$

for all p.

With this characterisation we have also established that the Schatten p-norms are invariant under unitary transformations.

b) Give the statement and proof for the Hölder inequality for Schatten p-norms.

The most important Schatten p-norms have other interesting expressions:

c) Show that the Schatten 2-norm or Frobenius fulfils

$$\|A\|_2^2 = \sum_{i,j=1}^{n} |A_{ij}|^2. \tag{4}$$

In general, one can define the operator norms induced by the ℓ_p-norms:

$$\|A\|_{\ell_p\to\ell_q} = \sup_{\|x\|_{\ell_p}=1} \|Ax\|_{\ell_q}. \tag{5}$$

d) What is the Schatten p-norm equal to $\|\cdot\|_{\ell_2\to\ell_2}$?

Another important properties of Schatten p-norms is *sub-multiplicativity*, $\|AB\|_p \leq \|A\|_p\|B\|_p$ for all p and $A, B \in \mathbb{C}^{n\times n}$. Sometimes the term *matrix norm* is exclusively used for sub-multiplicative norms on matrix spaces.

e) Show the sub-multiplicativity of the Schatten p-norms.

References

Banaszek, K. (2000, July). Optimal quantum teleportation with an arbitrary pure state. *Phys. Rev. A 62*(2), 024301.

Bennett, C. H., G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters (1993, March). Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels. *Phys. Rev. Lett. 70*(13), 1895–1899.