Freie Universität Berlin
Tutorials on Quantum Information Theory
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Problem Sheet 4
Teleportation, Schmidt decomposition and Schatten norms

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1. **Schmidt decomposition and purification**

In the lecture, you already saw the Schmidt decomposition of bipartite quantum states $|\Psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ as given by

$$|\Psi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_j} \left|\psi_j^1\right\rangle \left|\psi_j^2\right\rangle,$$

where $\{\left|\psi_j^i\right\rangle\}$ are orthonormal bases of $\mathcal{H}_i$.

In this exercise, we will study some useful properties and applications of the Schmidt decomposition. To begin with, let us look at states with the same Schmidt coefficients, that is

$$|\Psi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_j} \left|\psi_j^1\right\rangle \left|\psi_j^2\right\rangle, \quad |\Phi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_j} \left|\phi_j^1\right\rangle \left|\phi_j^2\right\rangle.$$

a) Show that $|\Psi\rangle$ and $|\Phi\rangle$ are related by a local unitary, i.e., a unitary of the form $U \otimes V$ with U and V unitary. Give that unitary explicitly.

Solution:

$$|\Psi\rangle = \left(\sum_j \left|\psi_j^1\right\rangle \left\langle\phi_j^1\right|\right) \otimes \left(\sum_j \left|\psi_j^2\right\rangle \left\langle\phi_j^2\right|\right) |\Phi\rangle$$

b) Show that any local unitary transformation leaves the Schmidt coefficients invariant.

Solution:

$$U \otimes V |\Psi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_j}\, U \left|\psi_j^1\right\rangle \otimes V \left|\psi_j^2\right\rangle,$$

but since U is a fixed unitary $U \left|\psi_j^i\right\rangle$ is still an orthogonal basis, hence we have a new state with the same Schmidt coefficients.

This gives rise to a nice interpretation of the Schmidt coefficients of a state in terms of entanglement (more soon!):

c) Determine the reduced density matrices $\rho_1 = \mathrm{Tr}_2 |\Psi\rangle\langle\Psi|$ and $\rho_2 = \mathrm{Tr}_1 |\Psi\rangle\langle\Psi|$. How can the Schmidt coefficients be interpreted? What are the Schmidt coefficients of the maximally entangled state?

Solution:

$$\rho_1 = \mathrm{Tr}_2\left[\sum_{ij}\sqrt{\lambda_i\lambda_j}\,|\psi_i^1\rangle\!\langle\psi_j^1|\,|\psi_i^2\rangle\!\langle\psi_j^2|\right]$$
$$= \sum_i \lambda_i\,|\psi_i^1\rangle\!\langle\psi_i^1|$$

The Schmidt coefficients are the eigenvalues of the reduced density matrix. the maximally entangled state has Schmidt coefficients $1/d$.

d) Use the Schmidt decomposition to show that *any* bipartite state $|\Psi\rangle$ can be expressed as

$$|\Psi\rangle = (X \otimes \mathbb{1})\,|\Omega\rangle\,,$$

where $|\Omega\rangle$ is a maximally entangled state.

Solution: Let $|\Psi\rangle$ have Schmidt decomposition as above. We then choose $|\Omega\rangle = \frac{1}{\sqrt{d}}\sum_i |\psi_i^2\rangle\,|\psi_i^2\rangle$, and $X = \sum_i \sqrt{d\lambda_i}\,|\psi_i^1\rangle\,\langle\psi_i^2|$.

The maximally entangled state is *invariant* under certain product unitaries $U \otimes V$.

e) What are the conditions on U and V for this to be the case?

Solution:

$$|\omega\rangle = U\otimes V\,|\omega\rangle \Leftrightarrow \frac{1}{\sqrt{d}}\sum_i U\,|i\rangle\,V\,|i\rangle = \frac{1}{\sqrt{d}}\sum_{ijk} U_{ji}\,|j\rangle\,V_{ki}\,|k\rangle = \frac{1}{\sqrt{d}}\sum_i |i\rangle\,|i\rangle$$

and hence $\sum_i U_{ji}V_{ki} = \sum_i U_{ji}V_{ik}^T = (UV^T)_{jk} = (VU^T)_{kj} = \delta_{jk}$. But this is the case *iff* $V^T = U^\dagger$ and hence $V = \overline{U}$.

Recall from the lecture that for any quantum state $\rho \in \mathcal{L}(\mathcal{H})$ there exists a pure quantum state $|\psi_\rho\rangle \in \mathcal{H}\otimes\mathcal{G}$ such that $\mathrm{Tr}_{\mathcal{G}}[|\psi_\rho\rangle\langle\psi_\rho|] = \rho$. The Schmidt decomposition is useful for explicitly constructing such purifications:

f) Give a purification of an arbitrary quantum state ρ in terms of its eigenvalues and eigenvectors.

Solution:

$$\rho = \sum_i \lambda_i\,|\psi_i\rangle\langle\psi_i| \Rightarrow |\psi_\rho\rangle = \sum_i \sqrt{\lambda_i}\,|\psi_i\rangle\,|\psi_i\rangle$$

g) Show that two purifications $|\psi_1^\rho\rangle$ and $|\psi_2^\rho\rangle$ of the same state ρ are related by a unitary transformation that acts on $\mathcal{G}$ only.

Solution: Let $|\psi_1^\rho\rangle$ and $|\psi_2^\rho\rangle$ be two purifications of the same state ρ, i.e. $\mathrm{Tr}_{\mathcal{G}}\,|\psi_1^\rho\rangle\langle\psi_1^\rho| = \rho = \mathrm{Tr}_{\mathcal{G}}\,|\psi_2^\rho\rangle\langle\psi_2^\rho|$.

We can write the Schmidt decomposition of $|\psi_1^\rho\rangle$ and $|\psi_2^\rho\rangle$ as

$$|\psi_1^\rho\rangle = \sum_i \sqrt{\lambda_i}\,|\psi_i\rangle\,|\psi_j^1\rangle$$
$$|\psi_2^\rho\rangle = \sum_i \sqrt{\lambda_i}\,|\psi_i\rangle\,|\psi_j^2\rangle\,,$$

which must hold for $\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$ since the eigendecomposition is unique and as we saw above, the Schmidt basis on the first component fully determines the reduced state.

But the two orthonormal bases $\{|\psi_j^1\rangle\}_j$, $\{|\psi_j^2\rangle\}_j$ on $\mathcal{G}$ are related via a unitary transformation that acts on $\mathcal{G}$ only.

2. **General teleportation schemes**

In the lecture you saw a teleportation scheme using a maximally entangled state shared by Alice and Bob. In this exercise we will generalise this setting to teleportation schemes with higher local dimensions.

We begin by reformulating the qubit teleportation scheme in terms of Bell-basis measurements. The Bell basis for two qubits is given by

$$|\Phi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), |\Phi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),$$
$$|\Phi_2\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), |\Phi_3\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

a) Show that the Bell basis can be prepared starting from $|\Phi_0\rangle$ using local Pauli operations only.

Solution:

$$\mathbb{1} \otimes Z |\Phi_0\rangle = |\Phi_1\rangle$$
$$\mathbb{1} \otimes X |\Phi_0\rangle = |\Phi_2\rangle$$
$$\mathbb{1} \otimes XZ |\Phi_0\rangle = |\Phi_3\rangle$$

b) Show that the scheme from the lecture is equivalent to the following one:

Alice and Bob share a maximally entangled state $|\Phi_0\rangle$, Alice prepares a state $|\omega\rangle = \alpha|0\rangle + \beta|1\rangle$, measures in the Bell basis and transmits her measurement result to Bob who applies the corresponding Pauli operator.

Solution: In the lecture, you saw the scheme in which Alice applies $(H\otimes\mathbb{1}^{\otimes 2})(CX\otimes\mathbb{1})$ to $|\psi\rangle|\Phi_0\rangle$ and then measures in the Z-basis. She then communicates her results, say a, b on the two registers to Bob, who applies $X^a Z^b$ as a correction to obtain $|\psi\rangle$ on his side.

The two schemes are equivalent via the identification of outcomes

$$00 \leftrightarrow 0, 10 \leftrightarrow 1, 01 \leftrightarrow 2, 11 \leftrightarrow 3,$$

where we used (a).

This reformulation generalises to a d-dimensional teleportation scheme in which Alice and Bob share a maximally entangled state $|\omega\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d}|ii\rangle$. As above the scheme is based on measuring in a maximally entangled orthonormal basis set $\{|\Psi_\alpha\rangle\}_{\alpha=1}^{d^2}$, i.e., an orthonormal basis for which $\mathrm{Tr}_1[|\Psi_\alpha\rangle\langle\Psi_\alpha|] = \mathbb{1}_d = \mathrm{Tr}_2[|\Psi_\alpha\rangle\langle\Psi_\alpha|]$.

There exist several constructions of linearly independent sets $\{U^\alpha\}_{\alpha=1}^{d^2}$ of d^2 trace-wise orthogonal unitary operators $U^\alpha \in U(d)$,

$$\mathrm{Tr}[U^{\alpha\dagger}U^\beta] = \mathrm{Tr}[U^{\alpha\dagger}U^\beta] = \delta_{\alpha\beta}\mathbb{1}$$

for all α and β. In the following, we just assume the existence of such a set.

c) Show that such a set $\{U^\alpha\}_{\alpha=1}^{d^2}$ gives rise to a maximally entangled basis set by setting

$$|\Psi_\alpha\rangle = U^\alpha \otimes \mathbb{1} |\omega\rangle .$$

Solution: Maximally entangled is clear.

This is a basis using

$$\langle \Psi_\alpha \mid \Psi_\beta\rangle = \langle\omega| U^{\alpha\dagger} U^\beta \otimes \mathbb{1} |\omega\rangle = \delta_{\alpha\beta}.$$

d) Use the completeness relation for $\{|\Psi_\alpha\rangle\}_\alpha$ to show that any such operator basis satisfies

$$\frac{1}{d}\sum_\alpha U^\alpha_{ij}\overline{U}^\alpha_{kl} = \delta_{ik}\delta jl. \tag{1}$$

Solution: First, note that $U^\alpha \otimes \mathbb{1} |\omega\rangle = \sum_{ijk} \frac{1}{\sqrt{d}} U^\alpha_{ij} |i\rangle\langle j| |kk\rangle = \sum_{ik} \frac{1}{\sqrt{d}} U^\alpha_{ik} |ik\rangle$. We then demand the completeness relation

$$\begin{aligned}\mathbb{1} = \sum_\alpha |\Psi_\alpha\rangle\langle\Psi_\alpha| &= \sum_\alpha U^\alpha \otimes \mathbb{1} |\omega\rangle\langle\omega| U^{\alpha\dagger} \otimes \mathbb{1} \\ &= \frac{1}{d}\sum_{ijkl} U^\alpha_{ij}\overline{U}^\alpha_{kl} |ij\rangle \langle kl| = \sum_{ij} |ij\rangle\langle ij| ,\end{aligned}$$

from which we conclude the claim.

e) Expand the basis states $|\Psi_\alpha\rangle$ in the computational product basis $\{|ij\rangle\}_{ij}$.

Solution: We have already seen $|\Psi_\alpha\rangle = U^\alpha \otimes \mathbb{1} |\omega\rangle = \sum_{ijk} \frac{1}{\sqrt{d}} U^\alpha_{ij} |i\rangle\langle j| |kk\rangle = \sum_{ik} \frac{1}{\sqrt{d}} U^\alpha_{ik} |ik\rangle$.

Now consider the setting in which Alice and Bob share the state $|\omega\rangle$ and Alice measures her part of the system in the basis $|\Psi_\alpha\rangle$.

f) Insert the resolution of the identity $\sum_\alpha |\Psi_\alpha\rangle\langle\Psi_\alpha|$ and use the result from (d) to derive the unitary corrections required in the d-dimensional teleportation scheme.

Solution: Using the expansion of $|\Psi_\alpha\rangle$ and $|\psi\rangle$ in the computational basis, we obtain

$$\begin{aligned}|\psi\rangle |\omega\rangle &= \sum_\alpha |\Psi_\alpha\rangle\langle\Psi_\alpha| |\psi\rangle |\omega\rangle \\ &= \frac{1}{d}\sum_{\alpha,ijkl} \overline{U}^\alpha_{ij}\psi_l |\Psi_\alpha\rangle \langle ij \mid l\rangle |kk\rangle \\ &= \frac{1}{d}\sum_{\alpha,ijk} \overline{U}^\alpha_{ij}\psi_i |\Psi_\alpha\rangle |j\rangle \\ &= \frac{1}{d}\sum_\alpha |\Psi_\alpha\rangle (U^\alpha)^\dagger |\psi\rangle ,\end{aligned}$$

where the last line is easily checked.

Further reading:

Bennett et al. (1993): The original teleportation paper.

Banaszek (2000): A d-dimensional teleportation scheme.

3. **Schatten p-norms**
On the last excercise sheet we have studied the ℓ_p-norms on vector spaces. The ℓ_p-norms have important cousins on matrix spaces, the Schatten p-norms. As they are important distant measures in quantum information, we study there different definitions and properties in this excerice.

One way to introduce the Schatten p-norm with $p \in [1, \infty)$ for a matrix $A \in \mathbb{C}^{n\times n}$ is

$$\|A\|_p := \left(\mathrm{Tr}\left[|A|^p\right]\right)^{\frac{1}{p}}, \tag{2}$$

where $|A| := \sqrt{A^\dagger A}$ is the matrix absolute value. Furthermore, the case $p = \infty$ is defined as the limit $\|A\|_\infty = \lim_{p\to\infty} \|A\|_p$.

These norms are related to the ℓ_p-norms of the eigenvalues (or more generally the singular values) of A.

a) Let A be a Hermitian matrix and let $\lambda = (\lambda_1, \dots, \lambda_n)$ be the vector of its eigenvalues. Show that

$$\|A\|_p = \|\lambda\|_{\ell_p} \tag{3}$$

for all p.

Solution: Let $A = U\Lambda U^\dagger$ be the eigenvalue decomposition of A. Recall that for Hermitian A it holds that $\sqrt{A^\dagger A} = U\,\mathrm{diag}(|\lambda_1|, \dots, |\lambda_n|)U^\dagger$ as can be easily checked by squaring the equation. Then,

$$\|A\|_p^p = \mathrm{Tr}\left[(A^\dagger A)^{\frac{p}{2}}\right] = \mathrm{Tr}\left[U\Lambda^p U^\dagger\right] = \mathrm{Tr}\,\Lambda^p = \sum_i \lambda_i^p = \|\lambda\|_{\ell_p}^p. \tag{4}$$

With this characterisation we have also established that the Schatten p-norms are invariant under unitary transformations.

b) Give the statement and proof for the Hölder inequality for Schatten p-norms.

Solution: There are different matrix version of Hölder's inequality: Let $1 \geq p \geq \infty$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$, then
Matrix Hölder I:

$$\|A^\dagger B\|_1 \leq \|A\|_p \|B\|_q. \tag{5}$$

Matrix Hölder II:

$$\left|\mathrm{Tr}\left[A^\dagger B\right]\right| \leq \|A\|_p \|B\|_q. \tag{6}$$

Just as a side remark, there is even the more general version that holds for every unitarily invariant norm $\|\cdot\|$.
Matrix Hölder III:

$$\|A^\dagger B\| \leq \|(A^\dagger)^p\|^{\frac{1}{p}} \|B^q\|^{\frac{1}{q}}. \tag{7}$$

The idea of the proof is to reduce the Matrix Hölder inequality to Hölder's inequality for ℓ_p-norms on vectors. To this end, we have to first establish the following inquality, the so-called von Neumann inequality. *Lemma "von Neumann-inequality":* Let A and B be two matrices and let $s(A)$ and $s(B)$ be the vector of singular values of A and B, respectively, ordered decreasingly. Then it holds that

$$|\operatorname{Tr}[AB]| \le \operatorname{Tr}|AB| \le \sum_i s_i(A)s_i(B). \tag{8}$$

The proof of this inequality is actually much more tricky than I have anticipated. If someone has a simpler or more elementary proof I am very keen to read it. (Seriously, please write me an email.) Some elegant ways to proof it is make use of the theory of majorisation and doubly stochastic matrices. To give you a flavor, here is a sketch of the argument following Bhatia's book on matrix analysis:

Proof sketch, the majorisation way: One can show that $s(AB)$ is weakly majorised[1] by the vector of the element-wise product $s(A)s(B)$, which is a stronger statement that implies the von Neumann inequality. For a matrix A, we denote by $\wedge^k A$ the linear map $A^{\otimes k}$ restricted to act only on vectors in $(\mathbb{C}^n)^{\otimes k}$ that are anti-symmetric under permuting the tensor factors. Using the min-max characterisation of singular values it is easy to see that $\| \wedge^k A\|_1 = \prod_{i=1}^k s_i$. Using the sub-multiplicativity of the spectral norm (see below) one concludes that $\prod_{i=1}^k s_i(AB) = \| \wedge^k AB\|_1 \le \| \wedge^k A\| \| \wedge^k B\| = \prod_{i=1}^k s_i(A)s_i(B)$. A function $\Phi : \mathbb{C}^n \to \mathbb{C}^m$ that preserves weak majorisation, i.e. x is weakly majorised by y implies that $\Phi(x)$ is weakly majorised by $\Phi(y)$, is called strongly isotone. The function acting as the exponential function on each component of a vector can be shown to be strongly isotone. Thus, we have for the component-wise logarithm that $\log x$ weakly majorised by $\log y$ implies that x is weakliy majorised by y. Using the preservation of majorisation when dropping the logarithm allows to conclude the desired statement that $s(AB)$ is weakly majorised by $s(A)s(B)$ from the relate statement about the partial products.

Slightly more direct proofs using doubly stochastic matrices were worked out by Mirsky. A more elementary proof was given R. D. Grigorieff in a note in '92. You can find it on his webpage.

Proof of matrix Hölder I & II.

With the help of the von Neumann inequality, it is easy to reduce matrix Hölder to the standard Hölder inequality for vectors:

$$\operatorname{Tr}|AB| \le |\langle s(A) \mid s(B)\rangle| \le \|s(A)\|_{\ell_p}\|s(B)\|_{\ell_q} = \|A\|_p\|B\|_q. \tag{9}$$

The second version follows from the first version by showing that

$$\left|\operatorname{Tr}\left[A^\dagger B\right]\right| \le \operatorname{Tr}\left|A^\dagger B\right|. \tag{10}$$

The most important Schatten p-norms have other interesting expressions:

c) Show that the Schatten 2-norm or Frobenius norm fulfils

$$\|A\|_2^2 = \sum_{i,j=1}^n |A_{ij}|^2. \tag{11}$$

[1] A n-dimensional vector x is weakly majorising a n-dimensional vector y if $\sum_{i=1}^k x_i \le \sum_{i=1}^k y_i$ for all $k \in \{1, 2, \dots, n\}$.

Solution:

$$\|A\|_2^2 = \operatorname{Tr} A^\dagger A = \sum_{i,j} \bar{A}_{ji} A_{ji} = \sum_{i,j} |A_{ij}|^2. \tag{12}$$

In general, one can define the operator norms induced by the ℓ_p-norms:

$$\|A\|_{\ell_p \to \ell_q} = \sup_{\|x\|_{\ell_p}=1} \|Ax\|_{\ell_q}. \tag{13}$$

d) What is the Schatten p-norm equal to $\|\cdot\|_{\ell_2\to\ell_2}$?

Solution:

$$\|A\|_{\ell_2\to\ell_2} = \sup_{\|x\|_{\ell_2}=1} \|Ax\|_{\ell_2} = \sup_{\|x\|_{\ell_2}=1} \sqrt{\langle x, A^\dagger A\, x\rangle} \tag{14}$$

$$= \sqrt{\lambda_{\mathsf{max}}(A^\dagger A)} = |\lambda_{\mathsf{max}}(A)|. \tag{15}$$

Another important properties of Schatten p-norms is *sub-multiplicativity*, $\|AB\|_p \leq \|A\|_p \|B\|_p$ for all p and $A, B \in \mathbb{C}^{n\times n}$. Sometimes the term *matrix norm* is exclusively used for sub-multiplicative norms on matrix spaces.

e) Show the sub-multiplicativity of the Schatten p-norms.

Solution: Using the min-max principle of the Rayleigh quotient, we first establish that $|\lambda_i(AB)| \leq \|A\|_\infty |\lambda_i(B)|$. Proof:

$$|\lambda_i(AB)| = \min_{U, \dim U = k} \max_{x\in U, \|x\|_{\ell_2}=1} |\langle x, ABx\rangle| \tag{16}$$

$$\leq \min_{U, \dim U = k} \max_{x\in U, \|x\|_{\ell_2}=1} \|A\|_\infty |\langle x, Bx\rangle| \tag{17}$$

$$= \|A\|_\infty |\lambda_i(B)|, \tag{18}$$

where we have used that $|\langle x, Ay\rangle| \leq \|A\|_\infty |\langle x, y\rangle|$, which follows from the operator norm definition of the spectral norm.

Now we have

$$\|AB\|_p = \left[\sum_i |\lambda_i(AB)|^p\right]^{\frac{1}{p}} \leq \|A\|_\infty \left[\sum_i |\lambda_i(B)|^p\right]^{\frac{1}{p}} \tag{19}$$

$$= \|A\|_\infty \|B\|_p \leq \|A\|_p \|B\|_p. \tag{20}$$

In the last step we have used the ordering of the p-norms inherited by the ordering of the ℓ_p-norms, in particular $\|A\|_\infty \leq \|A\|_p$ for all $p < \infty$.

Literatur

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