Freie Universität Berlin
**Tutorials on Quantum Information Theory**
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# Problem Sheet 5
# Quantum channels

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(*Hint:* While the first two excersises are more abstract the last one discusses simple examples and consists of hands-on calculations. If you get stuck somewhere on this sheet, try to jump ahead to the examples or vice versa.)

Let $\mathcal{X}$ and $\mathcal{Y}$ be two Hilbert spaces and $L(\mathcal{X})$ and $L(\mathcal{Y})$ denote the linear operators on the Hilbert spaces. In the lecture, you got to know quantum channels as those linear maps $T : L(\mathcal{X}) \to L(\mathcal{Y})$ such that the map $T \otimes \mathbb{1}_d$ maps quantum states to quantum states where $\mathbb{1}_d$ acts on an additional Hilbert space of arbitrary dimension $d$. You have seen that this amounts to requiring that quantum channels are linear maps that are completely positive and trace preserving (CPT). If not specified otherwise, on this sheet, $T$ will always denote such a quantum channel.

You also got to know various representations of quantum channels

- The *Kraus representation.* A map $T \in L(L(\mathcal{X}), L(\mathcal{Y}))$ is CPT *iff* there exist a set of linear operators $\{K_i\}_i$ with $K_i \in L(\mathcal{X}, \mathcal{Y})$ fulfiling $\sum_i K_i^\dagger K_i = \mathbb{1}$ such that

$$T(X) = \sum_i K_i X K_i^\dagger. \tag{1}$$

- The *Stinespring representation.* There exists an isometry $V \in L(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$ or, equivalently[1], an arbitrary reference state $|0\rangle \in \mathcal{Z}'$ and a corresponding unitary operator $U \in U(\mathcal{X} \otimes \mathcal{Z}')$ with $\mathcal{Y} \otimes \mathcal{Z} \cong \mathcal{X} \otimes \mathcal{Z}'$ such that

$$T(X) = \mathrm{Tr}_{\mathcal{Z}}[V X V^\dagger] = \mathrm{Tr}_{\mathcal{Z}}[U(X \otimes |0\rangle\langle 0|)U^\dagger]. \tag{2}$$

- The *Choi-Jamiołkowski representation.* $J(T) \in \mathcal{Y} \otimes \mathcal{X}$

$$J(T) \coloneqq (T \otimes \mathbb{1})\, |\omega\rangle\langle\omega|\,, \tag{3}$$

where $|\omega\rangle = \frac{1}{\sqrt{d}} \sum_i |ii\rangle$ is the maximally entangled state.

In this problem sheet, we will show the equivalence between those representations explicitly and consider some examples.

1. **On the Kraus representation of quantum channels**
The operational meaning of Kraus operators can be understood in the following setting in which, for simplicity, we restrict ourselves to quantum channels with the same input and output space $L(\mathcal{X})$. Suppose we apply a unitary $U$ to the joint system and environment in the state $\rho \otimes |0\rangle\langle 0| \in L(\mathcal{X} \otimes \mathcal{Z})$, where $|0\rangle \in \mathcal{Z}$ is some reference state, and then we measure system $\mathcal{Z}$ in the computational basis.
   a) Show that the action of the unitary on the joint system can be written as

$$U(\rho \otimes |0\rangle\langle 0|)U^\dagger = \sum_{kl} E_k \rho E_l^\dagger \otimes |k\rangle\,\langle l|\ ,$$

   with respect to the basis $\{|i\rangle\}_i$ on the second system.

[1] Here, we use that any isometry $V : \mathcal{X} \to \mathcal{X} \otimes \mathcal{Z}'$ can be written as $A = U(\mathbb{1} \otimes |0\rangle)$ with an arbitrary reference state $|0\rangle$ and a corresponding unitary $U \in U(\mathcal{X} \otimes \mathcal{Z}')$.

b) Now, we perform a von-Neumann measurement on $\mathcal{Z}$ in the same basis. Determine the post-measurement state conditioned on outcome $i$. What is the probability of obtaining outcome $i$?

c) Give the corresponding operational interpretation of the Kraus operators $E_k$ and the unitary $U$.

d) Now, suppose we want to implement a von-Neumann measurement on $\mathcal{X}$ in this manner. Characterize the unitaries $U \in U(\mathcal{X} \otimes \mathcal{Z})$ on the joint system that give rise to this situation. Give an example for the case of two qubits.

e) What do the operators $E_k$ have to satisfy such that one can reverse the channel after having (destructively) measured outcome $k$ on $\mathcal{Z}$?

Finally, we will show some properties of the Kraus representation

f) Let $\{K_i\}_{i=1}^N$ and $\{\tilde{K}_j\}_{i=1}^N$ be two sets of linear operators in $L(\mathcal{X}, \mathcal{Z})$ fulfilling the completeness relation of Kraus operators. Show that if the two sets are related by a unitary transformations $U \in U(N)$ such that $\tilde{K}_i = \sum_j U_{ij} K_j$, the channels represented by the sets coincide.

g) Show that all equal-sized Kraus representations of a given channel $T$ are related via a unitary transformation.

*Hint: Relate the Kraus representation two low-rank matrix factorisations of the Choi matrix.*

2. **Equivalence between representations of quantum channels**
Let us first show that the Choi-Jamiołkowski map $J : L(L(\mathcal{X}), L(\mathcal{Y})) \to L(\mathcal{Y} \otimes \mathcal{X})$ is a linear bijection between the CPT maps on the one hand and the set of quantum states on $\mathcal{Y} \otimes \mathcal{X}$ on the other hand.

a) Show that the inverse map can be defined by $\tilde{T}(X) = \mathrm{Tr}_{\mathcal{X}}[J(T)(\mathbb{1}_{\mathcal{Y}} \otimes X^T)]$ and makes $J$ a bijection.

Let $\rho_T \in \mathcal{Y} \otimes \mathcal{X}$ be the Choi-Jamiołkowski state corresponding to the quantum channel $T$.

b) Determine a set of Kraus operators representing $T$.

c) Determine a unitary $U_T$ representing $T$ via the Stinespring representation.

Now, let $U_T$ be a unitary representing $T$ in the Stinespring representation.

d) Determine the Choi-Jamiołkowski state representing $T$.

The rank of a quantum channel is defined as the rank of its Choi matrix.

e) Show that a quantum channel with rank $r$ can be represented as a Stinespring dilation using an auxiliary system of dimension $r$.

3. **Examples of quantum channels**
Now we are ready to look at some examples. The following maps are important so-called noise channels

$$F_\epsilon(A) := \epsilon XAX + (1-\epsilon)A$$
$$D_\epsilon(A) := \epsilon \,\mathrm{Tr}[A]\frac{\mathbb{1}}{d} + (1-\epsilon)A$$
$$A_\epsilon(A) := \epsilon \,\mathrm{Tr}[A]\,|0\rangle\langle 0| + (1-\epsilon)A,$$

where $\epsilon \in [0,1]$.

a) For each channel, show that it is CPT.

b) For each channel, give its Choi-Jamiołkowski state, a Kraus representation and a Stinespring representation.

   *Hint: It may help to consider* $\epsilon = 1$ *in a first step and then generalize to arbitrary* $\epsilon \in [0, 1]$.

c) Give a physical interpretation and a good name for each channel.