Freie Universität Berlin

Tutorials on Quantum Information Theory

Summer Term 2018

Problem Sheet 5
Quantum channels

Discussed in Tutorial: 31/05/2018

J. Eisert, D. Hangleiter, I. Roth

(*Hint:* While the first two excersises are more abstract the last one discusses simple examples and consists of hands-on calculations. If you get stuck somewhere on this sheet, try to jump ahead to the examples or vice versa.)

Let $\mathcal{X}$ and $\mathcal{Y}$ be two Hilbert spaces and $L(\mathcal{X})$ and $L(\mathcal{Y})$ denote the linear operators on the Hilbert spaces. In the lecture, you got to know quantum channels as those linear maps $T : L(\mathcal{X}) \to L(\mathcal{Y})$ such that the map $T \otimes \mathbb{1}_d$ maps quantum states to quantum states where $\mathbb{1}_d$ acts on an additional Hilbert space of arbitrary dimension d. You have seen that this amounts to requiring that quantum channels are linear maps that are completely positive and trace preserving (CPT). If not specified otherwise, on this sheet, T will always denote such a quantum channel.

You also got to know various representations of quantum channels

- The *Kraus representation.* A map $T \in L(L(\mathcal{X}), L(\mathcal{Y}))$ is CPT *iff* there exist a set of linear operators $\{K_i\}_i$ with $K_i \in L(\mathcal{X}, \mathcal{Y})$ fulfiling $\sum_i K_i^\dagger K_i = \mathbb{1}$ such that

$$T(X) = \sum_i K_i X K_i^\dagger. \tag{1}$$

- The *Stinespring representation.* There exists an isometry $V \in L(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$ or, equivalently[1], an arbitrary reference state $|0\rangle \in \mathcal{Z}'$ and a corresponding unitary operator $U \in U(\mathcal{X} \otimes \mathcal{Z}')$ with $\mathcal{Y} \otimes \mathcal{Z} \cong \mathcal{X} \otimes \mathcal{Z}'$ such that

$$T(X) = \mathrm{Tr}_{\mathcal{Z}}[V X V^\dagger] = \mathrm{Tr}_{\mathcal{Z}}[U(X \otimes |0\rangle\langle 0|)U^\dagger]. \tag{2}$$

- The *Choi-Jamiołkowski representation.* $J(T) \in \mathcal{Y} \otimes \mathcal{X}$

$$J(T) := (T \otimes \mathbb{1})\, |\omega\rangle\langle\omega|\,, \tag{3}$$

where $|\omega\rangle = \frac{1}{\sqrt{d}} \sum_i |ii\rangle$ is the maximally entangled state.

In this problem sheet, we will show the equivalence between those representations explicitly and consider some examples.

1. **On the Kraus representation of quantum channels**
The operational meaning of Kraus operators can be understood in the following setting in which, for simplicity, we restrict ourselves to quantum channels with the same input and output space $L(\mathcal{X})$. Suppose we apply a unitary U to the joint system and environment in the state $\rho \otimes |0\rangle\langle 0| \in L(\mathcal{X} \otimes \mathcal{Z})$, where $|0\rangle \in \mathcal{Z}$ is some reference state, and then we measure system $\mathcal{Z}$ in the computational basis.

   a) Show that the action of the unitary on the joint system can be written as

$$U(\rho \otimes |0\rangle\langle 0|)U^\dagger = \sum_{kl} E_k \rho E_l^\dagger \otimes |k\rangle\, \langle l|\ ,$$

   with respect to the basis $\{|i\rangle\}_i$ on the second system.

[1] Here, we use that any isometry $V : \mathcal{X} \to \mathcal{X} \otimes \mathcal{Z}'$ can be written as $A = U(\mathbb{1} \otimes |0\rangle)$ with an arbitrary reference state $|0\rangle$ and a corresponding unitary $U \in U(\mathcal{X} \otimes \mathcal{Z}')$.

Solution: We define $E_k = (\mathbb{1} \otimes \langle k|)U(\mathbb{1} \otimes |0\rangle)$ and check.

b) Now, we perform a von-Neumann measurement on $\mathcal{Z}$ in the same basis. Determine the post-measurement state conditioned on outcome i. What is the probability of obtaining outcome i?

Solution: Up to normalisation the post-measurement state is given by $\rho_i = E_k \rho E_k^\dagger$. The probability reads $p(i|\rho) = \mathrm{Tr}[(\mathbb{1} \otimes |i\rangle\langle i|)U(\rho \otimes |0\rangle\langle 0|)U^\dagger = \mathrm{Tr}[E_i^\dagger E_i \rho]$

c) Give the corresponding operational interpretation of the Kraus operators E_k and the unitary U.

Solution: The $E_k^\dagger E_k$ can be seen as elements of a POVM implemented on the first system by the von-Neumann measurement on the second system.

d) Now, suppose we want to implement a von-Neumann measurement on $\mathcal{X}$ via a global unitary and a von-Neumann measurement on $\mathcal{Z}$. Characterize the unitaries $U \in U(\mathcal{X} \otimes \mathcal{Z})$ on the joint system that give rise to this situation. Give an example for the case of two qubits.

Solution: They have to satisfy

$$[(\mathbb{1} \otimes \langle i|)U(\mathbb{1} \otimes |0\rangle)]^2 = (\mathbb{1} \otimes \langle i|)U(\mathbb{1} \otimes |0\rangle),$$
$$[(\mathbb{1} \otimes \langle i|)U(\mathbb{1} \otimes |0\rangle)]^\dagger = (\mathbb{1} \otimes \langle i|)U(\mathbb{1} \otimes |0\rangle).$$

An example is $CX = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes X$.

e) What do the operators E_k have to satisfy such that one can reverse the channel after having (destructively) measured outcome k on $\mathcal{Z}$?

Solution: Given k there must exist operators F_k such that

$$\frac{F_k E_k \rho E_k^\dagger F_k^\dagger}{\mathrm{Tr}[E_k \rho E_k^\dagger]} = \mathbb{1}_{\mathcal{X}}.$$

This is the case if only if the E_k are proportional to unitaries.

Finally, we will show some properties of the Kraus representation

f) Let $\{K_i\}_{i=1}^N$ and $\{\tilde{K}_j\}_{i=1}^N$ be two sets of linear operators in $L(\mathcal{X}, \mathcal{Z})$ fulfilling the completeness relation of Kraus operators. Show that if the two sets are related by a unitary transformations $U \in U(N)$ such that $\tilde{K}_i = \sum_j U_{ij} K_j$, the channels represented by the sets coincide.

Solution: We have

$$\tilde{T}(X) = \sum_j \tilde{K}_j X \tilde{K}_j^\dagger = \sum_{ijk} U_{ij} K_j X K_k^\dagger \overline{U}_{ik}$$
$$= \sum_{jk} \left(U_{ki}^\dagger U_{ij} \right) K_j X K_k^\dagger = \sum_{jk} \delta_{kj} K_j X K_k^\dagger,$$

where in the last equality we used the unitarity of U.

g) Show that all equal-sized Kraus representations of a given channel T are related via a unitary transformation.

Hint: Relate the Kraus representation of two low-rank matrix factorisations of the Choi matrix.

Solution: Let $\{K_i\}$ and $\{\tilde{K}_i\}$ be sets of Kraus operators. The Choi matrices of the corresponding quantum channel Φ can be expressed as

$$J(\Phi) = AA^\dagger = \tilde{A}\tilde{A}^\dagger \tag{4}$$

where the matrix A is given by

$$A = \big(\operatorname{vec} K_1, \operatorname{vec} K_2, \ldots, \operatorname{vec} K_N\big) \tag{5}$$

and $\tilde{A}$ defined analogously with the $\tilde{K}$s.

Recall from linear algebra that two low-rank factorisations $L_1 R_1 = L_2 R_2$ of the same matrix are always related by a invertible matrix G such that $L_1 = L_2 G$ and $R_1 = G^{-1} R_2$. For the case Hermitian matrices with $R_i = L_i^\dagger$, we conclude that G must be a unitary matrix.

Thus, there exists a unitary matrix $U(N)$ such that $\tilde{A} = AU$.

2. **Equivalence between representations of quantum channels**
Let us first show that the Choi-Jamiołkowski map $J : L(L(\mathcal{X}), L(\mathcal{Y})) \to L(\mathcal{Y} \otimes \mathcal{X})$ is a linear bijection between the CPT maps on the one hand and the set of quantum states on $\mathcal{Y} \otimes \mathcal{X}$ with partial-trace over $\mathcal{Y}$ is maximally mixed on the other hand.

   a) Show that the inverse map can be defined by $\tilde{T}(X) = \operatorname{Tr}_{\mathcal{X}}[J(T)(\mathbb{1}_{\mathcal{Y}} \otimes X^T)]$. and makes J a bijection as described above.

Solution: We have to show that the map J is both injective and surjective under suitable restrictions. For injectivity (1), we show that $\tilde{T} = T$, for surjectivity (2) we use a result from a previous sheet and the Kraus decomposition.

ad (1): We have that

$$\begin{aligned}
\tilde{T}(X) &= \sum_{ikl} (\mathbb{1} \otimes \langle i|)(T(|k\rangle\langle l|) \otimes |k\rangle\langle l|)(\mathbb{1} \otimes X^T)(\mathbb{1} \otimes |i\rangle) \\
&= \sum_{ikl} T(|k\rangle\langle l|) \langle i \mid k\rangle \langle l| X^T |i\rangle \\
&= \sum_{kl} T(|k\rangle\langle l|) \langle l| X^T |k\rangle \\
&= T(\sum_{kl} X_{kl} |k\rangle\langle l|) = T(X),
\end{aligned}$$

where the last line holds by the linearity of T.

ad (2): Let us now show that J is surjective. To this end, choose a state $\rho \in L(\mathcal{X} \otimes \mathcal{Y})$ with $\operatorname{Tr}_{\mathcal{Y}} \rho = \mathbb{1}/d$ with $d = \dim \mathcal{Y}$. We will show that there exists a quantum channel that has ρ as its Choi-Jamiołkowski isomorph. Express $\rho = \sum_i \lambda_i |t_i\rangle\langle t_i|$ in its eigenbasis.

We now make use of a fact proved on the last problem sheet, namely that for an arbitrary pure quantum state $|\psi\rangle$ we find an operator Y such that $|\psi\rangle = (Y \otimes \mathbb{1})|\Omega\rangle$. In particular, we can find operators K_i such that $\sqrt{d}(K_i \otimes \mathbb{1})|\Omega\rangle = \sqrt{\lambda_i}|t_i\rangle$. (Recall that this is just the inverse of the vectorisation map $\operatorname{vec} : L(\mathcal{X}) \cong \mathcal{X} \otimes \mathcal{X}^* \to \mathcal{X} \otimes \mathcal{X}$ that acts on a basis as $|i\rangle\langle j| \mapsto |i\rangle|j\rangle$.)

Due to the partial trace condition on ρ, the K_is fulfil

$$\sum_i K_i K_i^\dagger = d\,\mathrm{Tr}_{\mathcal{Y}}(K_i \otimes \mathbb{1})\,|\Omega\rangle\!\langle\Omega|\,(K_i^\dagger \otimes \mathbb{1}) \quad (6)$$

$$= \mathrm{Tr}_{\mathcal{Y}} \sum_i \lambda_i\,|t_i\rangle\!\langle t_i| \quad (7)$$

$$= \mathrm{Tr}_{\mathcal{Y}}\,\rho = \mathbb{1}/d, \quad (8)$$

where for the first step we inserted a funky looking identity $\mathbb{1} = d\,\mathrm{Tr}_{\mathcal{Y}}\,|\Omega\rangle\!\langle\Omega|$ between the Ks.

So the set $\{K_i\}$ satisfies the condition that we require of Kraus operators and, thus, define a CPT channel.

Let $\rho_T \in \mathcal{Y} \otimes \mathcal{X}$ be the Choi-Jamiołkowski state corresponding to the quantum channel T.

b) Determine a set of Kraus operators representing T.

Solution: Decompose $\rho_T = \sum_i \lambda_i\,|t_i\rangle\langle t_i|$ and let $K_i = \sqrt{\lambda_i}\mathrm{vec}^{-1}(|t_i\rangle)$, where vec^{-1} denotes the inverse map of vectorisation.

c) Determine a unitary U_T representing T via the Stinespring representation.

Solution: The isometry $V : L(\mathcal{X}) \to L(\mathcal{Y} \otimes \mathcal{Z})$ as in Def. (2) is given by $V = \sum_i K_i \otimes |i\rangle$, where $|i\rangle$ are orthonormal vectors in $\mathcal{Z}$ as is easily checked.

We now construct the unitary from the Stinespring representation as $U : L(\mathcal{X} \otimes \mathcal{Z}') \to L(\mathcal{Y} \otimes \mathcal{Z})$ by orthogonal completion (e.g. using Gram Schmidt) such that $U(\mathbb{1} \otimes |0\rangle) = V$ with $|0\rangle \in \mathcal{Z}'$.

Now, let U_T be a unitary representing T in the Stinespring representation.

d) Determine the Choi-Jamiołkowski state representing T.

Solution: We obtain the isometry $V = U_T(\mathbb{1} \otimes |0\rangle) = \sum_i K_i \otimes |i\rangle$ and then set $\rho_T = (\mathrm{vec}K_i)_i(\mathrm{vec}K_i)^\dagger$.

The rank of a quantum channel is defined as the rank of its Choi matrix.

e) Show that a quantum channel with rank r can be represented as a Stinespring dilation using an auxiliary system of dimension r.

Solution: We have $\rho_T = \sum_{i=1}^r \lambda_i\,|t_i\rangle\langle t_i|$, and hence $K_i = \sqrt{\lambda_i}\mathrm{vec}^{-1}(t_i), i = 1, \ldots, r$. Now define $V = \sum_{i=1}^r K_i \otimes |i\rangle$ as an isometry from $\mathcal{X}$ to $\mathcal{Y} \otimes \mathbb{C}^r$.

3. **Examples of quantum channels**

Now we are ready to look at some examples of quantum channels acting on qubits, i.e., $\mathcal{H} = \mathbb{C}^2$. The following maps are important so-called noise channels

$$F_\epsilon(A) := \epsilon XAX + (1-\epsilon)A$$

$$D_\epsilon(A) := \epsilon\,\mathrm{Tr}[A]\frac{\mathbb{1}}{d} + (1-\epsilon)A$$

$$A_\epsilon(A) := \epsilon\,\mathrm{Tr}[A]\,|0\rangle\langle 0| + (1-\epsilon)A,$$

where $\epsilon \in [0,1]$.

a) For each channel, show that it is CPT.

b) For each channel, give its Choi-Jamiołkowski state, a Kraus representation and a Stinespring representation.

Hint: It may help to consider $\epsilon = 1$ *in a first step and then generalize to arbitrary* $\epsilon \in [0, 1]$.

Solution: Let Id be the identity channel with $J(\mathrm{Id}) = |\Omega\rangle\langle\Omega|$. Then the Choi states are given by the convex combination of the channel with $\epsilon = 1$ and $J(\mathrm{Id})$, e.g. $J(F_\epsilon) = \epsilon J(F_1) + (1-\epsilon)J(\mathrm{Id})$. Now,

$$J(F_1) = \frac{1}{d}\sum_{ij} |i(i \oplus 1)\rangle \langle j(j \oplus 1)| \,, \quad J(D_1) = \frac{1}{d}\mathbb{1}, \quad J(A_1) = |0\rangle\langle 0| \otimes \mathbb{1},$$

where $\oplus$ denotes the addition modulo 2 a.k.a. xor.

We have the following possible Kraus representations of the channels

$$F(B) = XBX^\dagger, \quad D(B) = XBX + YBY + ZBZ + \mathbb{1}B\mathbb{1}$$
$$A(B) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} B \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} B \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$$

The isometries defining the Stinespring representation are: $V_F = X \otimes |0\rangle$, $V_D = \mathbb{1} \otimes |0\rangle + X \otimes |1\rangle + Y \otimes |2\rangle + Z \otimes |3\rangle$, likewise for V_A.

The corresponding unitaries are given, for example by $U_F = X \otimes |0\rangle\langle 0|$,

$U_D = \mathrm{SWAP}$ if the environment is prepared in the maximally mixed state.

c) Give a physical interpretation and a good name for each channel.

Solution: Bit-flip channel, depolarizing channel, amplitude damping channel.