Freie Universität Berlin
Tutorials on Quantum Information Theory
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Problem Sheet 7
Transforming and quantifying entanglement

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1. **Local operations and classical communication (LOCC).**

   At the heart of entanglement theory lies the notion of LOCC. To see why, imagine two parties that are a large distance apart from each other, say, Alice is in Berlin and Bob in New York. While they may obtain access to shared entanglement from a third party, it is unreasonable to assume that they are able to perform global operations on the state they share. On the other hand, it is perfectly conceivable that they transmit classical messages, for example, to communicate measurement results.

   In the LOCC paradigm, each party is allowed to measure and perform unitary operations on their part of the shared state, and communicate via a classical channel. While general LOCC protocols may involve several rounds of interaction (so Alice does something, sends the result to Bob, then Bob does something, who then communicates with Alice, and so forth) it is often sufficient to consider only a single round of interaction.

   So suppose, Alice and Bob share a state $|\psi\rangle$ with Schmidt decomposition $|\psi\rangle = \sum_i \sqrt{\lambda_i} \left|i^{(A)}\right\rangle \left|i^{(B)}\right\rangle$. We will now show that any measurement $\{M_j\}_j$ on Bob's side can be simulated as follows: Alice performs a measurement $\{N_j\}_j$ on her side, sends the result to Bob, who applies a corresponding unitary transformation.

   a) Expand M_j in the Schmidt basis $\{\left|i^{(B)}\right\rangle\}_i$ and define the measurement operator N_j in terms of the expansion coefficients. Determine the post-measurement state of Bob $|\psi_j\rangle$ (who performs $\{M_j\}$), and of Alice $|\phi_j\rangle$ (who performs $\{N_j\}$).
   b) Show that $|\phi_j\rangle$ is local-unitary equivalent to $|\psi_j\rangle$.
   c) Summarise the LOCC protocol.

2. **Majorisation and transforming quantum states by local unitaries.**

   In this problem we will look at the task of transforming a given copy of a pure bipartite quantum state $|\psi\rangle$ to another quantum state $|\phi\rangle$ using LOCC. The question is: Under which conditions is the transition $|\psi\rangle \xrightarrow{\text{LOCC}} |\phi\rangle$ possible?

   The key to the answer of this question is the concept of majorisation. We say that a real vector $x \in \mathbb{R}^n$ majorises $y \in \mathbb{R}^n$ ($x \succ y$) if for all $k = 1, \ldots, n$ $\sum_{j=1}^k x_j^\downarrow \geq \sum_{j=1}^k y_j^\downarrow$ and $\sum_{j=1}^n x_j^\downarrow = \sum_{j=1}^d y_j^\downarrow$. Here, $x^\downarrow$ denotes the sorted version of x, i.e., a permutation of the elements of x such that $x_1^\downarrow \geq x_2^\downarrow \geq \ldots \geq x_n^\downarrow$. So from now on, let $\sum_{j=1}^n x_j^\downarrow = \sum_{j=1}^d y_j^\downarrow$

   a) Show that $x \succ y$ if and only if for all $t \in \mathbb{R}$
   $$\sum_{j=1}^n \max(x_j - t, 0) \geq \sum_{j=1}^n \max(y_j - t, 0).$$
   b) Use the characterisation from (a) to show that the set $\{x : x \prec y\}$ is convex for any given y.

   One can now show that $x \prec y$ if and only if $x = \sum_j p_j \Pi_j y$ for a probability distribution p and permutation matrices Π_j. By Birkhoff's theorem, which lies at the heart of majorisation theory, that statement is equivalent to saying that $x \prec y$ if and only if $x = Dy$ for some doubly stochastic matrix D.

For two Hermitian operators $X, Y \in L(\mathbb{C}^d)$ we say that $X \prec Y$ if $\mathrm{spec}(X) \prec \mathrm{spec}(Y)$.

c) Show that $X \prec Y$ if and only if there exists a probability distribution p and unitary matrices U_j such that

$$X = \sum_j p_j U_j Y U_j^\dagger.$$

We are now ready to prove the main theorem:

Theorem 1 $|\psi\rangle \xrightarrow{LOCC} |\phi\rangle$ *if and only if* $\mathrm{spec}(\mathrm{Tr}_B[|\psi\rangle\langle\psi|]) \prec \mathrm{spec}(\mathrm{Tr}_B[|\phi\rangle\langle\phi|])$.

d) Show the forward direction using the result of Problem 1 you may assume that the transformation is effected by a measurement on Alice's side and a corresponding unitary on Bob's side. In other words, from Alice's point of view it must be the case that

$$M_j \,\mathrm{Tr}_B[|\psi\rangle\langle\psi|] M_j^\dagger = p_j \,\mathrm{Tr}_B[|\phi\rangle\langle\phi|].$$

Hint: Use the polar decomposition of $M_j \sqrt{\mathrm{Tr}_B[|\psi\rangle\langle\psi|]}$.

e) Now show the backward direction by proceeding analogously.

3. **Distilling and diluting entanglement.**

Now instead of being supplied with a single copy of an entangled state $|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ Alice and Bob have access to a large number of copies $|\psi\rangle^{\otimes m}$. We now ask two questions (that were already asked in the lecture): (1) How many copies of the Bell state $(|00\rangle + |11\rangle)/\sqrt{2}$, or *ebits* can be 'distilled' from $|\psi\rangle^{\otimes m}$? (2) Into how many 'less entangled states' $|\phi\rangle$ can $|\psi\rangle$ be diluted?

To begin with, recall the definition of ϵ-typical sequences: Given $\epsilon > 0$, a sequence $x = (x_1, x_2, \ldots, x_n)$ is called ϵ-typical with respect to a distribution p if

$$2^{-n(H(p)+\epsilon)} \le p(x_1)\cdots p(x_n) \le 2^{-n(H(p)-\epsilon)},$$

where $H(p) = -\sum_i p_i \log p_i$ is the Shannon entropy of p. Denote by $T(\epsilon, n)$ the set of length-n ϵ-typical sequences with respect to p. Also, recall the theorem of ϵ-typical sequences:

Theorem 2 *(i) Let $\epsilon > 0$. Then for any $\delta > 0\, \exists n \in \mathbb{N} : \Pr_{x_1,\ldots,x_n \sim p}[(x_1, \ldots, x_n) \in T(\epsilon, n)] \ge 1 - \delta$.*

(ii) $\forall \epsilon, \delta > 0\, \exists n \in \mathbb{N} : (1-\delta) 2^{n(H(p)-\epsilon)} \le |T(\epsilon, n)| \le 2^{n(H(p)+\epsilon)}$

We will now apply this theorem to the problem of diluting and distilling entanglement from $|\psi\rangle$. To this end, suppose that $|\psi\rangle = \sum_x \sqrt{p(x)}\, |x_A\rangle\, |x_B\rangle$ in Schmidt decomposition. Moreover, define by $|\phi_m\rangle$ the state obtained from $|\psi\rangle^{\otimes m}$ by omitting all terms that are not ϵ-typical and renormalising.

a) Show that the number of terms in $|\phi_m\rangle$ is at most $2^{m(S(\rho_\psi)+\epsilon)}$ where $\rho_\psi = \mathrm{Tr}_B[|\psi\rangle\langle\psi|]$.

Let us now look at the following entanglement dilution protocol: Alice and Bob share n Bell states. Alice locally prepares $|\phi_m\rangle$ and teleports one half of $|\phi_m\rangle$ to Bob.

b) How many Bell states are required such that after the dilution protocol Alice and Bob share $|\phi_m\rangle$.

c) Use Theorem 2 (i) to find a lower bound on the fidelity $|\langle\phi_m \mid \psi\rangle^{\otimes m}|^2$ for a suitably chosen m.

An entanglement concentration protocol proceeds along similar lines: Suppose Alice and Bob share n copies of $|\psi\rangle$. Alice performs a measurement that projects onto the ϵ-typical subspace of $|\psi\rangle$ to convert $|\psi\rangle^{\otimes} m$ into $|\phi_m\rangle$.

d) Determine an upper bound on the Schmidt coefficients of $|\phi_m\rangle$.

e) Determine n as a function of m such that the state $|\phi_m\rangle$ obtained from the projective measurement can be transformed into n ebits.

f) Show that the scaling of resources required for the distillation procedure is optimal.

Hint: Argue by means of a contradiction.

There are many more interesting issues that arise in the context of entanglement transformation, some of which we want to mention here but cannot go into here[1].

(i) There exist entangled states that cannot be distilled. These are precisely the positive-partial-transpose entangled states (Horodecki et al., 1998).

(ii) Some transformations between quantum states $|\psi\rangle$ and $|\phi\rangle$ using LOCC become possible only through a so-called catalyst state $|c\rangle$, that is, while $|\psi\rangle \xrightarrow{\text{LOCC}} |\phi\rangle$ is impossible, there exists a state $|c\rangle$ such that $|\psi\rangle\,|c\rangle \xrightarrow{\text{LOCC}} |\phi\rangle\,|c\rangle$ is possible.

(iii) For the original paper on concentrating entanglement see (Bennett et al., 1996).

References

Bennett, C. H., H. J. Bernstein, S. Popescu, and B. Schumacher (1996, April). Concentrating partial entanglement by local operations. *Phys. Rev. A 53*(4), 2046–2052.

Horodecki, M., P. Horodecki, and R. Horodecki (1998, June). Mixed-State Entanglement and Distillation: Is there a “Bound” Entanglement in Nature? *Physical Review Letters 80*(24), 5239–5242.

[1]... because the sheet is already a bit long...